

Mathematics for Quantum Field Theory

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Problem Set 0

Groups, representations, and Lie theory

Purpose and instructions

This sheet progresses from group homomorphisms and quotients to rotations, Lorentz transformations, and the finite-dimensional irreducible representations of $SU(2)$. The problems build on one another and are intended for independent study and discussion. You may use the standard linear-algebra facts stated in the questions. All Lorentz transformations act on column vectors, with $\eta = \text{diag}(1, -1, -1, -1)$; spatial rotations and boosts are interpreted actively.

0.1 Homomorphisms, kernels, and images

Let $f : G \rightarrow H$ be a group homomorphism, and recall $\ker f = \{g \in G : f(g) = e_H\}$ and $\text{im } f = \{f(g) : g \in G\}$. You may use the subgroup test: a nonempty subset K of a group is a subgroup if $xy^{-1} \in K$ for all $x, y \in K$.

- a) Prove that $f(e_G) = e_H$.
- b) Prove that $f(g^{-1}) = f(g)^{-1}$ for every $g \in G$.
- c) Use the subgroup test to prove that $\ker f \leq G$ and $\text{im } f \leq H$.
- d) Prove that f is injective if and only if $\ker f = \{e_G\}$.

0.2 The quotient-group theorem

Let $N \leq G$, and let $G/N = \{gN : g \in G\}$ be the set of left cosets. Recall that N is normal, written $N \triangleleft G$, if $gNg^{-1} = N$ for all $g \in G$.

- a) Prove that

$$gN = g'N \iff g^{-1}g' \in N.$$

- b) Assume that $N \triangleleft G$. Prove that

$$(gN)(hN) := (gh)N$$

is independent of the chosen representatives g and h .

- c) Verify the group axioms for this product. Identify the identity and the inverse of gN .
- d) Prove the converse: if the displayed product is well defined on the set of left cosets, then N is normal in G .

0.3 The double cover $SU(2) \rightarrow SO(3)$

Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and let H_0 be the real vector space of traceless Hermitian 2×2 matrices. Here $A^\dagger$ denotes the conjugate transpose. Repeated spatial indices are summed from 1 to 3, and ϵ_{ijk} is totally antisymmetric with $\epsilon_{123} = 1$; spatial indices are raised using δ_{ij} .

- a) Prove the Pauli identities

$$\sigma_i \sigma_j = \delta_{ij} \mathbf{1} + i \epsilon_{ij}^{k} \sigma_k, \quad \frac{1}{2} \text{tr}(\sigma_i \sigma_j) = \delta_{ij}.$$

Deduce that $\mathbf{x} \mapsto \hat{\mathbf{x}} := \mathbf{x} \cdot \boldsymbol{\sigma}$ is an isometry $\mathbb{R}^3 \rightarrow H_0$ when $\langle X, Y \rangle = \frac{1}{2} \text{tr}(XY)$.

- b) Show that every $A \in SU(2)$ has a unique expression

$$A = a_0 \mathbf{1} - i \mathbf{a} \cdot \boldsymbol{\sigma}, \quad a_0 \in \mathbb{R}, \quad \mathbf{a} \in \mathbb{R}^3, \quad a_0^2 + |\mathbf{a}|^2 = 1.$$

Conclude that $SU(2)$ is diffeomorphic to S^3 and is connected.

- c) Define $\pi(A)$ by

$$A(\mathbf{x} \cdot \boldsymbol{\sigma}) A^\dagger = (\pi(A) \mathbf{x}) \cdot \boldsymbol{\sigma}.$$

Prove that $\pi : SU(2) \rightarrow SO(3)$ is a continuous homomorphism, and derive

$$\pi(A)_{ij} = \frac{1}{2} \text{tr}(\sigma_i A \sigma_j A^\dagger).$$

- d) For $|\mathbf{n}| = 1$, set

$$A(\mathbf{n}, \theta) = \exp\left(-\frac{i\theta}{2} \mathbf{n} \cdot \boldsymbol{\sigma}\right).$$

Starting from the Pauli identities, prove Rodrigues' formula

$$\begin{aligned} \pi(A(\mathbf{n}, \theta)) \mathbf{x} &= (\mathbf{n} \cdot \mathbf{x}) \mathbf{n} + \cos \theta [\mathbf{x} - (\mathbf{n} \cdot \mathbf{x}) \mathbf{n}] \\ &\quad + \sin \theta \mathbf{n} \times \mathbf{x}. \end{aligned}$$

- e) Prove that $\ker \pi = \{\pm \mathbf{1}\}$. You may use the axis-angle description of rotations to prove that π is surjective. Show directly that the induced map $[A] \mapsto \pi(A)$ is a well-defined bijective homomorphism, and hence that

$$SU(2)/\{\pm \mathbf{1}\} \simeq SO(3).$$

- f) Compute $A(\mathbf{n}, 2\pi)$ and $A(\mathbf{n}, 4\pi)$. The path $\theta \mapsto A(\mathbf{n}, \theta)$ projects under π to rotations about $\mathbf{n}$. Explain why the rotation path closes at 2π , whereas this path in $SU(2)$ first returns to $\mathbf{1}$ at 4π .

0.4 Minkowski orthonormality and the Lorentz group

Let $\eta = \text{diag}(1, -1, -1, -1)$, write $x \cdot y = x^\top \eta y$, and put $q(x) = x \cdot x$. The Lorentz group $O(1, 3)$ consists of the invertible real-linear maps preserving q . A nonzero vector is called timelike, spacelike, or null when $q(x) > 0$, $q(x) < 0$, or $q(x) = 0$, respectively.

Reminder (polarization). A symmetric bilinear form is recovered from its quadratic form by the polarization identity

$$x \cdot y = \frac{1}{2}(q(x+y) - q(x) - q(y)).$$

Thus knowing all squared lengths also determines all pairings; positivity of the form is not required.

- a) Verify the polarization identity by expanding $q(x+y)$. Use it and the linearity of $\Lambda : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ to prove that $q(\Lambda x) = q(x)$ for every x if and only if $(\Lambda x) \cdot (\Lambda y) = x \cdot y$ for every x, y .

- b) Prove

$$O(1, 3) = \{\Lambda \in \text{Mat}(4, \mathbb{R}) : \Lambda^\top \eta \Lambda = \eta\}.$$

Show also that the matrix equation itself forces Λ to be invertible, and that every $\Lambda \in O(1, 3)$ has determinant ± 1 .

- c) Call an ordered basis (b_0, b_1, b_2, b_3) Minkowski-orthonormal if $b_\mu \cdot b_\nu = \eta_{\mu\nu}$. Prove that for any two such bases there is a unique Lorentz transformation sending the first to the second.
- d) Deduce that a real 4×4 matrix is Lorentz if and only if its columns form a Minkowski-orthonormal basis. Explain why Lorentz transformations preserve the timelike, spacelike, and null classes.

0.5 Rotations, boosts, parity, and time reversal

The proper orthochronous Lorentz transformations are, by definition,

$$\text{SO}^+(1, 3) = \{\Lambda \in O(1, 3) : \det \Lambda = 1, \Lambda^0_0 > 0\}.$$

Here “proper” means determinant $+1$, and “orthochronous” means preserving the future time direction, as will be established in Problem 0.6. For $R \in O(3)$ write $\tilde{R} = \text{diag}(1, R)$, and define

$$B_3(\chi) = \begin{pmatrix} \cosh \chi & 0 & 0 & \sinh \chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \chi & 0 & 0 & \cosh \chi \end{pmatrix}.$$

Physical meaning of rapidity. Rapidity $\chi \in \mathbb{R}$ is the dimensionless hyperbolic angle parametrizing a boost, just as an ordinary angle parametrizes a rotation. We use χ to reserve η for the Minkowski metric. With $x^0 = ct$, the displayed matrix is an *active* boost: for $\chi > 0$, it sends a particle at rest to motion in the positive z direction. Its velocity is related to rapidity by $\beta = v/c = \tanh \chi$, or $\chi = \operatorname{artanh} \beta$. A passive change to coordinates of a frame moving with velocity $+v$ uses the inverse $B_3(-\chi)$ instead.

- a) Prove that $\tilde{R} \in O(1, 3)$, and determine when $\tilde{R} \in \operatorname{SO}^+(1, 3)$.
- b) Verify directly that $B_3(\chi) \in \operatorname{SO}^+(1, 3)$ and prove

$$B_3(\chi_1)B_3(\chi_2) = B_3(\chi_1 + \chi_2), \quad B_3(\chi)^{-1} = B_3(-\chi).$$

Apply the boost to the rest worldline $t \mapsto (ct, 0, 0, 0)$ and show that the transformed velocity satisfies $v/c = \tanh \chi$. Deduce $\gamma = (1 - \beta^2)^{-1/2} = \cosh \chi$ and $\gamma\beta = \sinh \chi$. Use the boost composition law to derive the collinear velocity-addition formula

$$\beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}.$$

- c) Let

$$P = \operatorname{diag}(1, -1, -1, -1), \quad T = \operatorname{diag}(-1, 1, 1, 1).$$

Verify that $P, T \in O(1, 3)$ and that $\{\mathbf{1}, P, T, PT\}$ is an abelian subgroup of order four.

- d) Starting from $\Lambda^T \eta \Lambda = \eta$, prove

$$\Lambda^T = P\Lambda^{-1}P = T\Lambda^{-1}T.$$

Deduce $\Lambda\eta\Lambda^T = \eta$ and hence that a matrix is Lorentz if and only if its rows form a Minkowski-orthonormal basis.

0.6 The four components of the Lorentz group

Define the future timelike cone and the orthochronous Lorentz group by

$$V_+ = \{x \in \mathbb{R}^{1,3} : x \cdot x > 0, x^0 > 0\}, \quad O^\uparrow(1, 3) = \{\Lambda \in O(1, 3) : \Lambda^0_0 > 0\}.$$

- a) Show that $|\Lambda^0_0| \geq 1$ for every $\Lambda \in O(1, 3)$.
- b) Prove that $\Lambda \in O^\uparrow(1, 3)$ if and only if $\Lambda(V_+) = V_+$. Deduce that $O^\uparrow(1, 3)$ is a subgroup.
- c) Recall that $\operatorname{SO}^+(1, 3) = O^\uparrow(1, 3) \cap SL(4, \mathbb{R})$. Prove that every $\Lambda \in O(1, 3)$ lies in exactly one of

$$\operatorname{SO}^+(1, 3), \quad \operatorname{PSO}^+(1, 3), \quad \operatorname{TSO}^+(1, 3), \quad \operatorname{PTSO}^+(1, 3).$$

Record $\det \Lambda$ and the sign of Λ^0_0 for each set.

- d) Let $\Lambda \in \text{SO}^+(1, 3)$ and write $\Lambda e_0 = (u, \mathbf{v})$. Choose $\chi \geq 0$ and $R_1 \in \text{SO}(3)$ so that

$$u = \cosh \chi, \quad \mathbf{v} = R_1(0, 0, \sinh \chi).$$

Show that $C = B_3(-\chi)\tilde{R}_1^{-1}\Lambda$ fixes e_0 , and prove that every proper Lorentz transformation fixing e_0 is $\tilde{R}_2 = \text{diag}(1, R_2)$ for some $R_2 \in \text{SO}(3)$.

- e) Conclude the decomposition

$$\Lambda = \tilde{R}_1 B_3(\chi) \tilde{R}_2.$$

Use it to prove that $\text{SO}^+(1, 3)$ is connected and that the four sets in part c) are exactly the four connected components of $O(1, 3)$.

0.7 The spin cover of the connected Lorentz group

Introduce $\sigma_0 = \mathbf{1}$ and identify $x \in \mathbb{R}^{1,3}$ with

$$\hat{x} = x^0 \mathbf{1} + x^i \sigma_i = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}.$$

- a) Prove that this identifies $\mathbb{R}^{1,3}$ real-linearly with the Hermitian 2×2 matrices, and verify $\det \hat{x} = (x^0)^2 - |\mathbf{x}|^2$.
- b) For $S \in \text{SL}(2, \mathbb{C})$, define $\Lambda(S)$ by

$$\widehat{\Lambda(S)x} = S \hat{x} S^\dagger.$$

Prove that $\Lambda(S) \in O(1, 3)$ and that $\Lambda(S_1 S_2) = \Lambda(S_1) \Lambda(S_2)$.

- c) For $S_\chi = \text{diag}(e^{\chi/2}, e^{-\chi/2})$, calculate $S_\chi \hat{x} S_\chi^\dagger$ and prove $\Lambda(S_\chi) = B_3(\chi)$.
- d) You may use polar decomposition. Write $S = UH$ with $U \in \text{SU}(2)$ and H positive-definite Hermitian of determinant one. Show that $H = e^h$ for a traceless Hermitian matrix h , and use this to prove that $\text{SL}(2, \mathbb{C})$ is connected. Deduce that $\Lambda(S) \in \text{SO}^+(1, 3)$ for every S .
- e) Prove $\ker \Lambda = \{\pm \mathbf{1}\}$. Hint: first use $\hat{x} = \mathbf{1}$, then use the Pauli matrices.
- f) Show that the restriction to $\text{SU}(2)$ fixes x^0 and agrees with the map $\pi : \text{SU}(2) \rightarrow \text{SO}(3)$ from Problem 0.3 on the spatial components.
- g) Use Problem 0.6 and the surjectivity $\text{SU}(2) \rightarrow \text{SO}(3)$ to prove that $\Lambda : \text{SL}(2, \mathbb{C}) \rightarrow \text{SO}^+(1, 3)$ is surjective. As in Problem 0.3, prove that the induced quotient map is an isomorphism, giving

$$\text{SL}(2, \mathbb{C}) / \{\pm \mathbf{1}\} \simeq \text{SO}^+(1, 3).$$

0.8 The finite-dimensional irreducible representations of $SU(2)$

Let $\rho : SU(2) \rightarrow GL(V)$ be a smooth finite-dimensional complex representation on $V \neq 0$. Invariant subspaces in this problem are always complex subspaces. Write $\mathfrak{sl}(2, \mathbb{C}) = \{Z \in M_2(\mathbb{C}) : \text{tr } Z = 0\}$ and use the conventions

$$\begin{aligned}\mathfrak{su}(2) &= \{X \in \mathfrak{sl}(2, \mathbb{C}) : X^\dagger = -X\}, & X_a &= -\frac{i}{2}\sigma_a, \\ d\rho(X_a) &= -iJ_a, & J_\pm &= J_1 \pm iJ_2.\end{aligned}$$

The differential is $d\rho(X) = \left. \frac{d}{dt} \right|_{t=0} \rho(e^{tX})$. You may use the following facts: $d\rho$ is a real-linear Lie-algebra homomorphism, $\rho(e^X) = e^{d\rho(X)}$, and every element of a connected matrix Lie group is a finite product of exponentials of Lie-algebra elements. Together with connectedness of $SU(2)$ from Problem 0.3, these facts let you pass between the group and its Lie algebra.

a) For $Z \in \mathfrak{sl}(2, \mathbb{C})$ prove the unique decomposition

$$Z = X + iY, \quad X = \frac{Z - Z^\dagger}{2}, \quad Y = \frac{Z + Z^\dagger}{2i}, \quad X, Y \in \mathfrak{su}(2).$$

If $\tau : \mathfrak{su}(2) \rightarrow \text{End}_{\mathbb{C}}(V)$ is a real-linear Lie-algebra homomorphism, prove that

$$\tau_{\mathbb{C}}(X + iY) = \tau(X) + i\tau(Y)$$

defines a complex-linear representation of $\mathfrak{sl}(2, \mathbb{C})$. Prove that a complex subspace is τ -invariant if and only if it is $\tau_{\mathbb{C}}$ -invariant. Deduce that τ is irreducible (respectively, completely reducible) if and only if $\tau_{\mathbb{C}}$ is.

b) In $\mathfrak{sl}(2, \mathbb{C})$ put

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Verify

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

For $H = \tau_{\mathbb{C}}(h)$, $E = \tau_{\mathbb{C}}(e)$, and $F = \tau_{\mathbb{C}}(f)$, obtain the same commutators. Show that

$$h = 2iX_3, \quad e = iX_1 - X_2, \quad f = iX_1 + X_2,$$

and hence, for $\tau = d\rho$,

$$H = 2J_3, \quad E = J_+, \quad F = J_-.$$

c) Assume $(\tau_{\mathbb{C}}, V)$ is irreducible. Choose an eigenvalue λ of H with maximal real part and an eigenvector $v_0 \neq 0$. Prove $Ev_0 = 0$. For $v_k = F^k v_0$, prove by induction

$$Hv_k = (\lambda - 2k)v_k, \quad Ev_k = k(\lambda - k + 1)v_{k-1} \quad (k \geq 1).$$

- d) Show that the span of the nonzero v_k is invariant and hence is all of V . Show that these vectors are linearly independent, and use finite dimensionality to show that some $n \geq 0$ satisfies $v_n \neq 0$ and $v_{n+1} = 0$. Apply E to v_{n+1} to prove $\lambda = n \in \mathbb{N}_0$. Conclude that $\dim V = n + 1$.
- e) Conversely, for each $n \in \mathbb{N}_0$, on $V_n = \mathbb{C}^{n+1}$ with basis $v_0, \dots, v_n$ and $v_{-1} = v_{n+1} = 0$, define

$$Hv_m = (n - 2m)v_m, \quad Ev_m = m(n + 1 - m)v_{m-1}, \quad Fv_m = v_{m+1}.$$

Check all three commutator relations, including the endpoints, and prove that V_n is irreducible. Conclude that every finite-dimensional irreducible $\mathfrak{sl}(2, \mathbb{C})$ -module is equivalent to exactly one V_n .

- f) Let U_n be the homogeneous degree- n polynomials in $z = (z_1, z_2)^\top$ and define

$$(\rho_n(A)p)(z) = p(A^{-1}z), \quad A \in SL(2, \mathbb{C}).$$

Prove that this is a smooth left representation and that $\dim U_n = n + 1$. Derive

$$(d\rho_n(X)p)(z) = -(Xz) \cdot \nabla p(z),$$

where $(Xz) \cdot \nabla = \sum_{r=1}^2 (Xz)_r \partial_r$ is a complex-bilinear contraction (no conjugation). For $H = d\rho_n(h)$, $E = d\rho_n(e)$, and $F = d\rho_n(f)$, deduce

$$H = -z_1 \partial_1 + z_2 \partial_2, \quad E = -z_2 \partial_1, \quad F = -z_1 \partial_2.$$

Prove that

$$\Phi(v_m) = \frac{(-1)^m}{(n-m)!} z_1^m z_2^{n-m}$$

intertwines V_n with U_n . Deduce that $\rho_n|_{SU(2)}$ is irreducible and that every finite-dimensional irreducible representation of $SU(2)$ is equivalent to exactly one such restriction.

- g) Put

$$q_m = \frac{(-1)^m z_1^m z_2^{n-m}}{\sqrt{m!(n-m)!}}, \quad 0 \leq m \leq n,$$

and set $q_{-1} = q_{n+1} = 0$. Declare $q_0, \dots, q_n$ orthonormal. Compute the actions of H, E, F and prove $H^\dagger = H$, $E^\dagger = F$. Show that every $X \in \mathfrak{su}(2)$ has the form

$$X = ia h + b e - \bar{b} f, \quad a \in \mathbb{R}, \quad b \in \mathbb{C},$$

and deduce that $d\rho_n(X)$ is anti-Hermitian. Use generation by exponentials to prove that $\rho_n|_{SU(2)}$ is unitary. Finally, prove that its invariant Hermitian inner product is unique up to a positive scalar.

h) Set $j = n/2$, $\mu = j - m$, and $|j, \mu\rangle = q_m$. Derive the normalized ladder formulas

$$\begin{aligned} J_3 |j, \mu\rangle &= \mu |j, \mu\rangle, \\ J_+ |j, \mu\rangle &= \sqrt{(j - \mu)(j + \mu + 1)} |j, \mu + 1\rangle, \\ J_- |j, \mu\rangle &= \sqrt{(j + \mu)(j - \mu + 1)} |j, \mu - 1\rangle. \end{aligned}$$

Here kets outside $-j \leq \mu \leq j$ are understood to be zero. Use $\mathbf{J}^2 = J_3^2 + \frac{1}{2}(J_+ J_- + J_- J_+)$ to prove $\mathbf{J}^2 = j(j + 1)\mathbf{1}$.

- i) Use Problem 0.3 to prove that an $SU(2)$ representation descends to $SO(3)$ if and only if $-\mathbf{1}$ acts trivially. Compute $\rho_n(-\mathbf{1})$ and show that descent occurs precisely when n is even, equivalently when j is integral or $n + 1$ is odd.

Some problems develop proofs of results discussed in the lectures. Problems 0.3–0.8 are translated and adapted from Giovanni Felder, *Mathematische Methoden der Physik II* (ETH Zürich, © 2018). Source and adaptation: CC BY-NC-SA 4.0.